**Question One (15 Marks)**

a) Use integration by parts to find $\int_1^2 x \ln x \, dx$ **2**

b) Use partial fractions to show that $\int \frac{4}{x^4 - 1} dx = \ln\left(\frac{x-1}{x+1}\right) - 2\tan^{-1} x + C$ **3**

c) Use the substitution $2\tan\theta = x + 2$ to show that **4**

$$\int \frac{dx}{\left(x^2 + 4x + 8\right)^{3/2}} = \frac{x+2}{4\sqrt{x^2 + 4x + 8}} + C.$$

d) (i) Establish the following relation **6**

$$\int \sin^n x \, dx = -\frac{1}{n}\sin^{n-1} \cos x + \frac{n-1}{n}\int \sin^{n-2} x \, dx \ .$$

(ii) Hence evaluate $\int_0^{\pi/2} \sin^6 x \, dx$.

**6**

**Question Two (15 Marks) Start a new booklet**

a) (i) On an argand diagram, shade in the region $\mathcal{R}$ containing all the points representing the complex number *z* such that

$$1 < |z| < 2 \text{ and } \frac{\pi}{4} < \arg z < \frac{\pi}{2}$$

(ii) In $\mathcal{R}$ mark with a dot the point *K* representing a complex number *z*. Clearly indicate on your diagram the points *M*, *N*, *P* and *Q*. representing the complex numbers $\overline{z}, -z, \frac{1}{z}, 2z$ respectively.

*Q2 ctd …p. 2*

**Question Two (continued)**

b) The hyperbola **H** has equation $xy = 16$. **9**

(i) Sketch this hyperbola and indicate on your diagram the positions and co-ordinates of all points at which the curve intersects the axes of symmetry.

(ii) $P\left(4p, \frac{4}{p}\right)$, where $p > 0$, and $Q\left(4q, \frac{4}{q}\right)$, where $q > 0$, are two distinct arbitrary points on **H**. Find the equation of the chord *PQ*.

(iii) Prove that the equation of the tangent at *P* is $x + py^2 = 8p$.

(iv) The tangents at *P* and *Q* intersect at *T*. Find the co-ordinates of *T*.

(v) The chord *PQ* produced passes through the point $N(0,8)$.

Find the equation of the locus of *T*.

(vi) Give a geometrical description of this locus.

**Question Three (15 Marks) Start a new booklet**

a) Two functions are defined as follows:

$$f(x) = \frac{1}{1+x^2} \qquad\qquad g(x) = 5 + e^{-2x}$$ **8**

(i) For each function, state its domain and range, and whether it is odd, even or neither odd nor even.

(ii) Find the first derivative of each function.

(ii) Sketch the graph of each function.

b) Let $f(x) = \frac{2-x}{x}$. On separate diagrams, sketch the graphs of the following functions. For each graph label any asymptotes. **7**

(i) $y = f(x)$

(ii) $y = f(|x|)$

(iii) $y = e^{f(x)}$

(iv) $y^2 = f(x)$

Discuss the behaviour of the curve of (iv) at $x = 2$.

*Q4 …p3*

**Question Four (15 Marks) Start a new booklet**

a) Use the substitution $u = x^{\frac{1}{2}}$ to evaluate $\int_0^2 \frac{dx}{2+\sqrt{x}}$. **4**

b) Alex decides to go bungy-jumping. This involves being tied to a bridge at a point *O* by an elastic cable of length *l* metres, and then falling vertically from rest from this point. **11**

After Alex fee-falls *l* metres, he is slowed down by the cable, which exerts a force, in newtons, of *Mgk* times the distance greater than *l* that he has fallen (where *M* is his mass in kilograms, $g$ m/s$^2$ is the constant acceleration due to gravity, and *k* is constant).

Let *x* m be the distance Alex has fallen, and let $v$ m/s be his speed at *x*. You may assume that his acceleration is given by $\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{1}{2}v^2\right)$.

(i) Show that $\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{1}{2}v^2\right) = g$ when $x \le l$ and $\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{1}{2}v^2\right) = g - gk(x-l)$ when $x > l$.

(ii) Show that $v^2 = 2gl$ when Alex first passes $x = l$ .

(iii) Show that $v^2 = 2gx - kg(x-l)^2$ for $x > l$.

(iv) Show that Alex's fall is first halted at $x = l + \frac{1}{k} + \sqrt{\frac{2l}{k} + \left(\frac{1}{k}\right)^2}$ .

(v) Suppose $\frac{1}{k} = \frac{l}{4}$ . Show that *O* must be at least 2 *l* metres above any obstruction on Alex's path.

**Question Five (15 Marks) Start a new booklet**

a) (i) Find the modulus and argument of the complex number $1+i$.

(ii) Use the binomial expansion of $(1+i)^n$, where $n$ is a positive integer, to show that **6**

(α) $1-{}^nC_2+{}^nC_4-\ldots=2^{n/2}\cos\dfrac{n\pi}{4}$

(β) ${}^nC_1-{}^nC_3+{}^nC_5\ldots=2^{n/2}\sin\dfrac{n\pi}{4}$

b) (i) If $\alpha, \beta, \gamma$ are zeros of the polynomial $x^3-px^2+qx-r$, prove $\beta^2\gamma^2+\gamma^2\alpha^2+\alpha^2\beta^2=q^2-2pr$. **5**

(ii) Show also that the polynomial whose roots are $\beta^2\gamma^2, \gamma^2\alpha^2, \alpha^2\beta^2$ is $x^3+(2pr-q^2)x^2+(p^2r^2-2qr^2)x-r^4=0$.

c) Show that the locus specified by $3|z-4-4i|=|z-12-12i|$ is a circle. Write down its radius and the coordinates of its centre. Draw a neat sketch of the circle. **4**

**Question Six (15 Marks) Start a new booklet**

a) Using the method of cylindrical shells, find the volume of the solid obtained by rotating the region bounded by the curves $y = x$ and $y = \sqrt{x}$ about the line $x = 1$. **5**

b) A drinking glass having the form of a right circular cylinder of radius *a* and height *h*, is filled with water. The glass is slowly tilted over, spilling water out of it, until it reaches the position where the water's surface bisects the base of the glass. Figure 1 shows this position. **10**

Figure 1

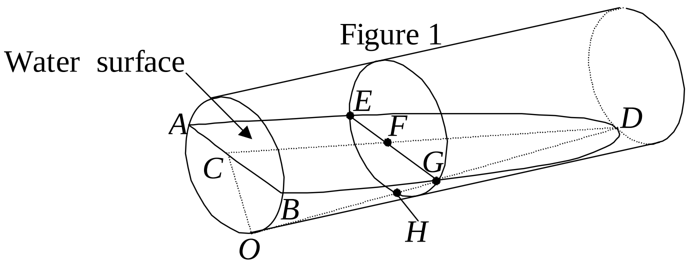



Figure 2

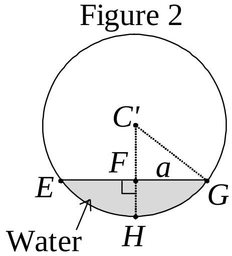



Figure 3

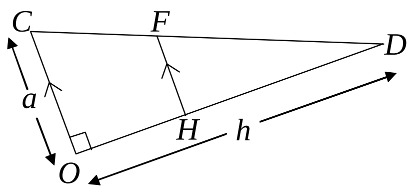



Figure 3 shows the section *COD* of the tilted glass.

Note: $FH \parallel CO$, $CO = a$, and $OD = h$.

In figure 1, *AB* is the diameter of the circular base with centre *C*, *O* is the lowest point on the base, and *D* is the point where the water's surface touches the rim of the glass. Figure 2 shows a cross-section of the tilted glass parallel to its base. The centre of this circular section is $C'$ and *EFG* shows the water level. The section cuts the lines *CD* and *OD* of figure 1 in *F* and *H* respectively.

**Question Six (continued)**

(i) Use figure 3 to show that $FH = \frac{a}{h}(h - x)$, where $OH = x$.

(ii) Use figure 2 to show that $C'F = \frac{ax}{h}$ and $\angle HC'G = \cos^{-1}\left(\frac{x}{h}\right)$.

(iii) Use (ii) to show that the area of the shaded segment $EGH$ is

$$a^2\left[\cos^{-1}\left(\frac{x}{h}\right) - \left(\frac{x}{h}\right)\sqrt{1 - \left(\frac{x}{h}\right)^2}\right].$$

(iv) Given that $\int \cos^{-1}\theta \, d\theta = \theta\cos^{-1}\theta - \sqrt{1-\theta^2}$, find the volume of water in the tilted glass of figure 1.

**Question Seven (15 Marks) Start a new booklet**

a) Show that if $r > \frac{q^2}{4}$, then $\int \frac{dx}{x^2 + qx + r} = \frac{2}{\sqrt{4r - q^2}} \tan^{-1}\left(\frac{2x + q}{\sqrt{4r - q^2}}\right) + C$ **4**

b) The ellipse **E** : $\left(\frac{x}{5}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$ has foci $S(4,0)$ and $S'(-4,0)$. **11**

(i) Sketch the ellipse **E** indicating its foci and its directrices.

(ii) Show that the tangent at $P(x_1, y_1)$ on the ellipse **E** has equation $9x_1x + 25y_1y = 225$.

(iii) The line joining $P(x_1, y_1)$ to $Q(x_2, y_2)$ passes through *S*. Show that $4(y_2 - y_1) = x_1y_2 - x_2y_1$.

(iv) It is also known that $Q(x_2, y_2)$ lies on **E**. Show that the tangents at *P* and *Q* on the ellipse intersect on the directrix corresponding to *S*.

(v) Find the equation of the normal to **E** at *P* and determine under what circumstances, if any, it passes through *S* or $S'$.

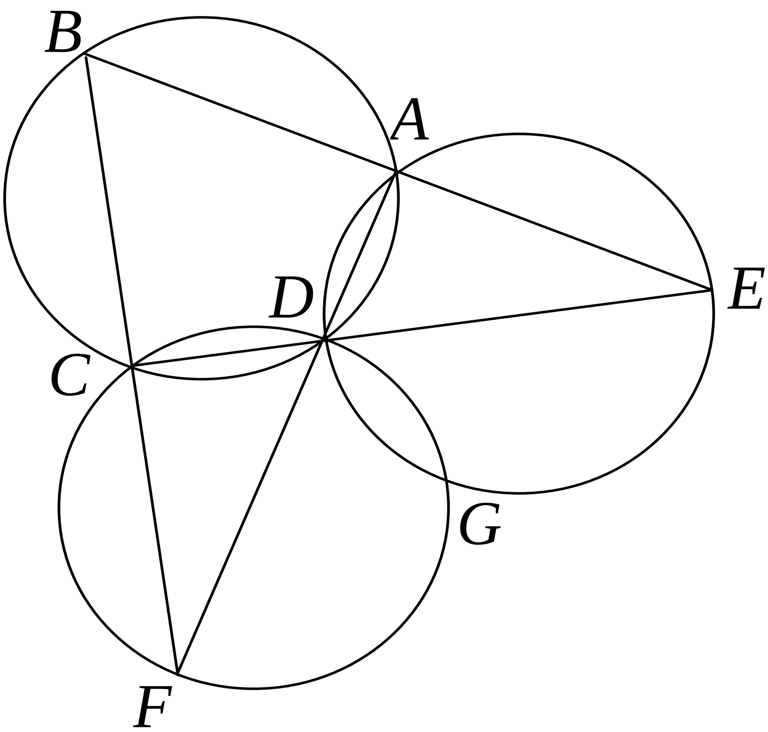
*Q8 …p.7*

**<u>Question Eight (15 Marks) Start a new booklet</u>**

a) Prove that $\lim_{x \to 0} \frac{1-\sqrt{1-x}}{x} = \frac{1}{2}$ **2**

b) A coin is tossed six times. What is the probability that there will be more tails on the first three of these six throws then there will be on the last three throws? **3**

c) *ABCD* is a cyclic quadrilateral. *BA* and *CD* are *both* produced to intersect at *E*. *BC* and *AD* produced intersect at *F*. The circles *EAD*, *FCD* intersect at *G* as well as *D*. Prove that the points *E*, *G* and *F*are collinear. **3**




*Q8 ctd …p.8*

**Question Eight (continued)**

d) Newton's method may be used to determine numerical approximations to the real roots of the equation $x^3 = 2$. Let $x_1 = 2$ be the first approximation and $x_2, x_3, x_4, \ldots x_n, \ldots$ be a series of estimations obtained by iterative applications of Newton's method. **7**

(i) Show that $x_{n+1} = \frac{2}{3}\left(x_n + \frac{1}{x_n^2}\right)$.

(ii) Show algebraically that

$$x_{n+1} - \sqrt[3]{2} = \frac{\left(x_n - \sqrt[3]{2}\right)^2 \left(2x_n + \sqrt[3]{2}\right)}{3x_n^2}.$$

(iii) Given that $x_n > \sqrt[3]{2}$, show that $x_{n+1} - \sqrt[3]{2} < \left(x_n - \sqrt[3]{2}\right)^2$.

(iv) Show that $x_{12}$ and $\sqrt[3]{2}$ agree to at least 267 decimal places.

**End of Paper**

Ext2 Trial 2006 

1)
(a) $\left[\frac{1}{2}x^2 \ln x\right]_1^2 - \int_1^2 \frac{1}{2}x^2 \cdot \frac{1}{x}\,dx$

$= 2\ln 2 - \frac{1}{2}\int_1^2 x\,dx$

$= 2\ln 2 - \frac{1}{2}\left[\frac{1}{2}x^2\right]_1^2$

$= 2\ln 2 - \frac{1}{4}(4-1)$

$= 2\ln 2 - \frac{3}{4}$

(b) $\frac{4}{x^4-1} \equiv \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{x+1}$

$4 \equiv (Ax+B)(x-1)(x+1) + C(x^2+1)(x+1) + D(x^2+1)(x-1)$

Let $x=1$, $4 = 4C$, $C=1$.

Let $x=-1$, $4=-4D$, $D=-1$

$A+C+D=0$ (coefficient of $x^3$)

$\therefore A=0$

Let $x=0$, $4=-B+C-D$

$\therefore B=-2$

$\therefore \int \frac{4}{x^4-1}\,dx = \int\left(\frac{-2}{x^2+1} + \frac{1}{x-1} - \frac{1}{x+1}\right)dx$

$= -2\tan^{-1}x + \log(x-1) - \log(x+1) + C$

$= \log\left(\frac{x-1}{x+1}\right) - 2\tan^{-1}x + C$

(c) $x+2 = 2\tan\theta$

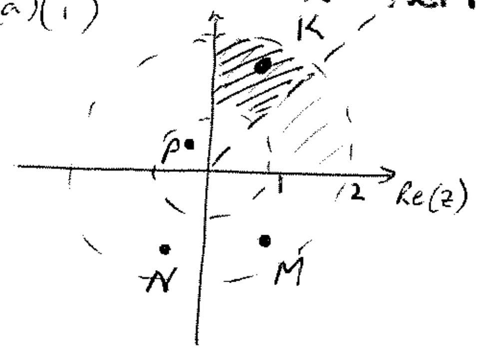
$dx = 2\sec^2\theta\,d\theta$

$\int \frac{dx}{(x^2+4x+8)^{3/2}} = \int \frac{dx}{((x+2)^2+4)^{3/2}}$

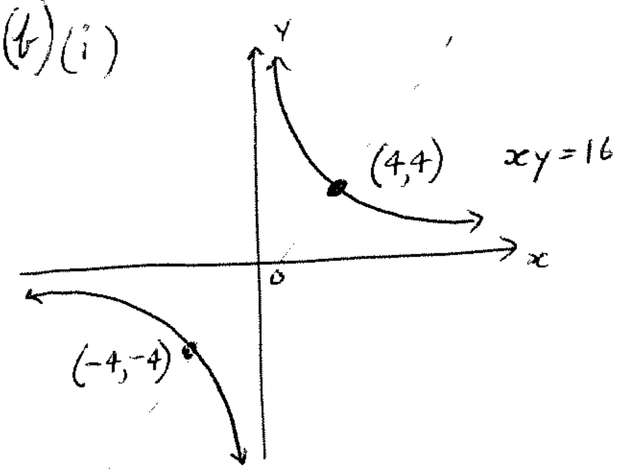
$= \int \frac{2\sec^2\theta\,d\theta}{(4\tan^2\theta+4)^{3/2}}$

$= \frac{1}{4}\int \frac{\sec^2\theta\,d\theta}{(\tan^2\theta+1)^{3/2}} = \frac{1}{4}\int \frac{\sec^2\theta\,d\theta}{\sec^3\theta} = \frac{1}{4}\int \cos\theta\,d\theta$

(2) (a)(i)




(b)(i)




(ii) $m_{PQ} = \dfrac{\frac{4}{p} - \frac{4}{q}}{4p - 4q}$

$= \dfrac{4(q-p)}{pq \cdot 4(p-q)}$

$= -\dfrac{1}{pq}$

Equation PQ:

$y - \frac{4}{p} = -\frac{1}{pq}(x - 4p)$

$x + pqy = 4(p+q)$

(iii) tangent at P = limiting position of PQ as $P \to Q$

ie.

$x + p \cdot py = 4(p+p)$

$x + p^2 y = 4p^2$

or calculus/differentiation etc.

(iv) tangent at Q:

$x + q^2 y = 4q^2$ —— ①

tangent at P:

$x + p^2 y = 4p^2$ —— ②

Solve ①, ② simultaneously

$T\left(\dfrac{8pq}{p+q}, \dfrac{8}{p+q}\right)$

(v) Sub (0, 8) into (ii).

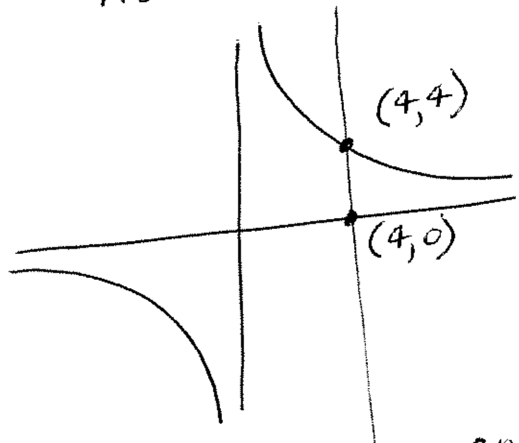
$0 + pq\,8 = 4(p+q)$

$2pq = (p+q)$

$\therefore T\left(4, \dfrac{8}{p+q}\right)$

$\therefore$ locus of T is $x = 4$

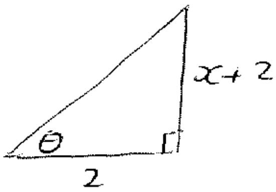
(vi) Consider when T can lie on $x = 4$.




two tangents can only intersect on $x = 4$ for $0 < y < 4$.

$\therefore$ locus is $x = 4$ for $0 < y < 4$.

$$= \frac{1}{4}\sin\theta + C$$




$$= \frac{1}{4}\cdot\frac{x+2}{\sqrt{(x+2)^2+4}} + C$$

$$= \frac{x+2}{\sqrt{x^2+4x+8}} + C$$

(d)
(i) $I_n = \int \sin^n x \, dx$

$$= \int \sin^{n-1}x \cdot \sin x \, dx$$

$$= \sin^{n-1}x \cdot -\cos x + \int \cos x \cdot (n-1)\sin^{n-2}x \cdot \cos x \, dx$$

$$= -\sin^{n-1}x\cos x + (n-1)\int (1-\sin^2 x)\sin^{n-2}x \, dx$$

$$= -\sin^{n-1}x\cos x + (n-1)I_{n-2} - (n-1)I_n$$

$$I_n(1+n-1) = -\sin^{n-1}x\cos x + (n-1)I_{n-2}$$

$$I_n = \frac{-1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}I_{n-2}$$

(ii) $I_6 = \int_0^{\frac{\pi}{2}} \sin^6 x \, dx$

$$= -\frac{1}{6}\Big[\sin^5 x\cos x\Big]_0^{\frac{\pi}{2}} + \frac{5}{6}I_4$$

$$= \frac{5}{6}\left(-\frac{1}{4}\Big[\sin^3 x\cos x\Big]_0^{\frac{\pi}{2}} + \frac{3}{4}I_2\right)$$

$$= \frac{5}{6}\cdot\frac{3}{4}\left(-\frac{1}{2}\Big[\sin x\cos x\Big]_0^{\frac{\pi}{2}} + \frac{1}{2}I_0\right)$$

$$= \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} dx$$

$$= \frac{5}{16}\cdot\frac{\pi}{2}$$

$$= \frac{5\pi}{32}$$

3) (a) $f(x) = \frac{1}{1+x^2}$

(i)

Domain: $x \in \mathbb{R}$

Range: $0 < y \leq 1$

$f(x)$ is even

$g(x) = 5 + e^{-2x}$

Domain: $x \in \mathbb{R}$

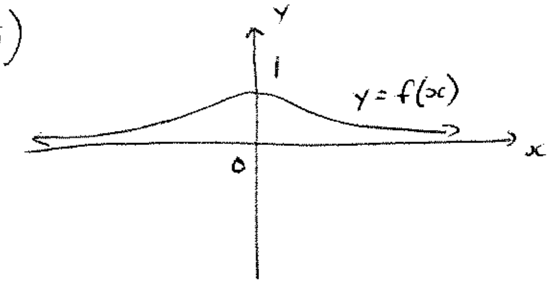
Range: $y > 5$

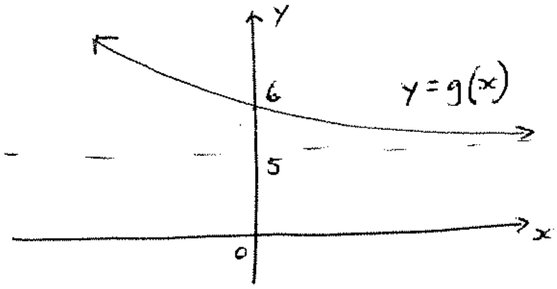
$g(x)$ is neither odd nor even.

(ii) $f'(x) = \frac{-2x}{(1+x^2)^2}$

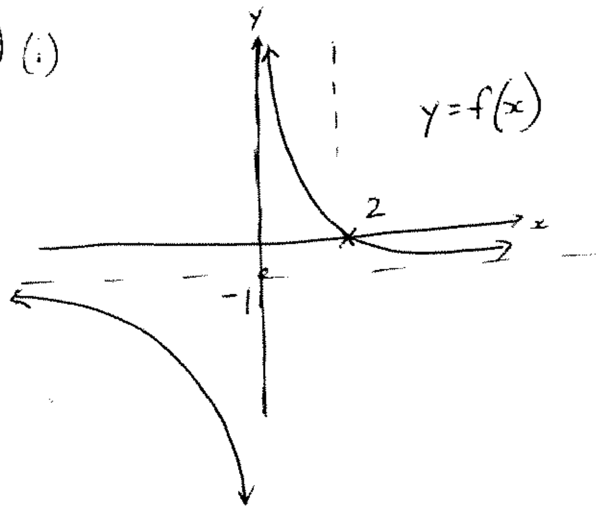
$g'(x) = -2e^{-2x}$

(iii)

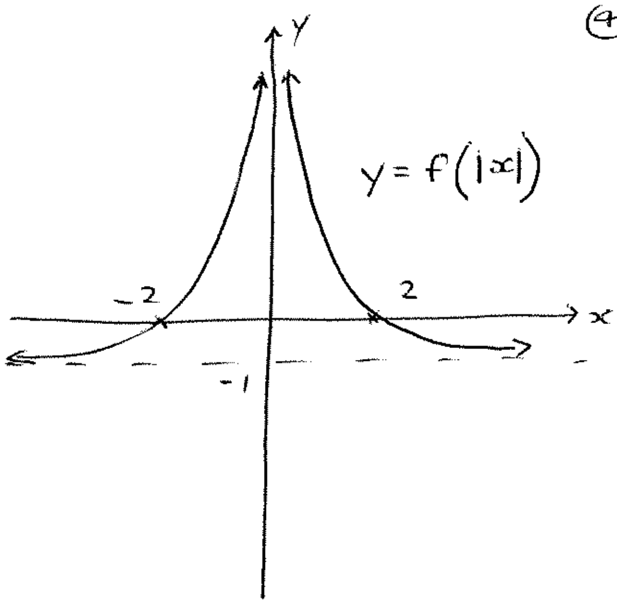


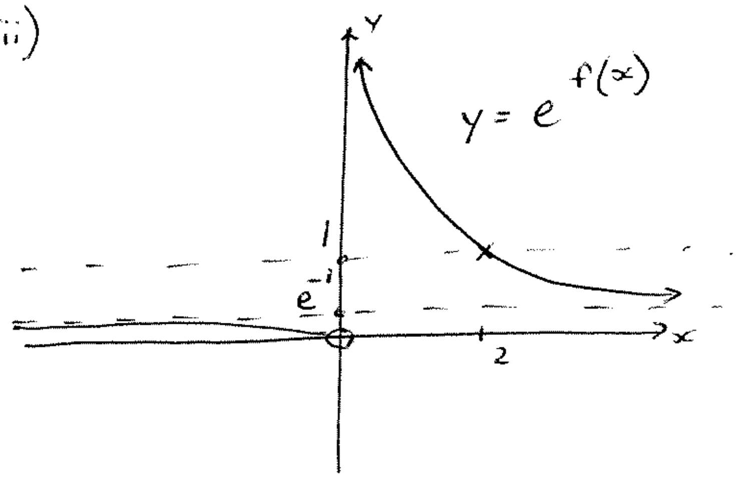

(b) (i)



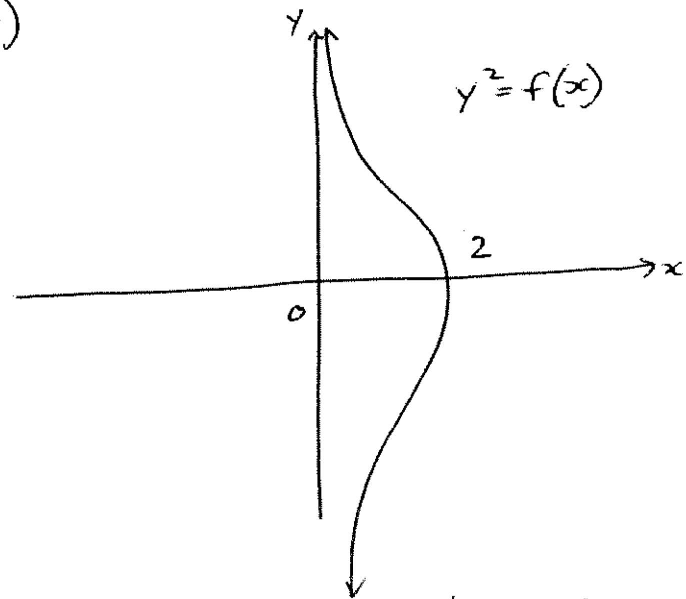

(ii)




(iii)




(iv)




at $x = 2$, $f'(x)$ is discontinuous

ie. $f(x)$ has a vertical tangent at $x = 2$

7) (a) $u = x^{\frac{1}{2}}$

$\therefore du = \frac{1}{2\sqrt{x}} \cdot dx$

$2u\,du = dx$

$x=0,\ u=0$

$x=2,\ u=\sqrt{2}$

$$\int_0^2 \frac{dx}{2+\sqrt{x}} = \int_0^{\sqrt{2}} \frac{2u\,du}{2+u}$$

$$= 2\int_0^{\sqrt{2}} \frac{u+2-2}{u+2}\,du$$

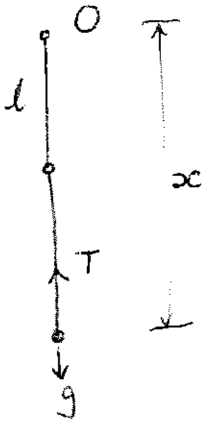
$$= 2\int_0^{\sqrt{2}} \left(1 - \frac{2}{u+2}\right) du$$

$$= 2\Big[u - 2\log(u+2)\Big]_0^{\sqrt{2}}$$

$$= 2\left(\sqrt{2} - 2\log(\sqrt{2}+2) - (0 - 2\log 2)\right)$$

$$= 2\left(\sqrt{2} - 2\log\left(\frac{2}{\sqrt{2}+2}\right)\right)$$

(b)




Let downwards be positive

(i) When $x \le l$, $F = Mg$, $T=0$

$\therefore Mg = M\ddot{x}$

$g = \ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

When $x > l$, $F = Mg - T$

$\therefore F = Mg - Mgk(x-l)$

$\therefore M\ddot{x} = Mg - Mgk(x-l)$

$\therefore \ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= g - gk(x-l)$

(ii) $x \le l$, $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = g$

$\therefore \frac{1}{2}v^2 = gx + c$

$x = 0,\ v=0 \ \therefore c = 0$

$v^2 = 2gx$, at $x = l$

$v^2 = 2gl$.

(iii) $x > l$

$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = g - gk(x-l)$

$\frac{1}{2}v^2 = gx - \frac{gk}{2}(x-l)^2 + c_1$

$x = l,\ v = \sqrt{2gx}$

$\therefore gx = gx - 0 + c_1,\ \therefore c_1 = 0$

$\therefore \frac{1}{2}v^2 = gx - \frac{gk}{2}(x-l)^2$

$v^2 = 2gx - gk(x-l)^2$

(iv) $v = 0$

$\therefore 2gx - gk(x-l)^2 = 0$

$2x - k(x^2 - 2xl + l^2) = 0$

$kx^2 - 2(kl+1)x + kl^2 = 0$

$$\therefore x = \frac{2(kl+1) \pm \sqrt{4(kl+1)^2 - 4k^2}}{2k}$$

$$= \frac{kl+1 \pm \sqrt{k^2l^2 + 2kl + 1 - k}}{k}$$

$$= l + \frac{1}{k} \pm \sqrt{\frac{2kl}{k^2} + \frac{1}{k^2}}$$

Now, 1st halted $\Rightarrow$ maximum $\Rightarrow$ beyond $l +$

$$\therefore x = l + \frac{1}{k} + \sqrt{\frac{2l}{k} + \frac{1}{k^2}}$$

$$\therefore x = l + \frac{l}{4} + \sqrt{\frac{2l^2}{4} + \frac{l^2}{16}}$$
$$= l + \frac{l}{4} + \sqrt{\frac{9l^2}{16}}$$
$$= l + \frac{l}{4} + \frac{3l}{4}$$
$$= 2l$$

$\therefore$ Fall is first halted at $x = 2l$.

)(a)(i) $|1+i| = \sqrt{2}$

$Arg(1+i) = \frac{\pi}{4}$

(ii) $(1+i)^n = 1 + {}^nC_1 i + {}^nC_2 i^2 + \dots$
$$= \sum_{r=0}^{n} {}^nC_r i^r$$
$$= 1 + {}^nC_2 i^2 + {}^nC_4 i^4 + \dots + \left({}^nC_1 i + {}^nC_3 i^3 + {}^nC_5 i^5 + \dots\right)$$

$+ \quad (1+i)^n = \left(\sqrt{2}\, cis\left(\frac{\pi}{4}\right)\right)^n$
$$= 2^{\frac{n}{2}} cis \frac{n\pi}{4} \quad \text{by De Moivre's Theorem.}$$
$$= 2^{\frac{n}{2}} \cos\frac{n\pi}{4} + i \cdot 2^{\frac{n}{2}} \sin\frac{n\pi}{4}$$

$$2^{\frac{n}{2}} \cos\frac{n\pi}{4} = 1 + {}^nC_2 i^2 + {}^nC_4 i^4 + \dots$$
$$= 1 - {}^nC_2 + {}^nC_4 - \dots$$

$$2^{\frac{n}{2}} \sin\frac{n\pi}{4} = {}^nC_1 i + {}^nC_3 i^3 + {}^nC_5 i^5 + \dots$$

$$2^{\frac{n}{2}} \sin\frac{n\pi}{4} = {}^nC_1 + {}^nC_3 i^2 + {}^nC_5 i^4 + \dots$$
$$= {}^nC_1 - {}^nC_3 + {}^nC_5 - \dots$$

(b) (i) $\alpha+\beta+\gamma = p$

$\alpha\beta+\alpha\gamma+\beta\gamma = q$

$\alpha\beta\gamma = r$

Now,

$$(\alpha\beta+\alpha\gamma+\beta\gamma)^2 = q^2$$
$$= \alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2+2(\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2)$$
$$\therefore \alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2 = q^2-2\alpha\beta\gamma(\alpha+\beta+\gamma)$$
$$= q^2-2rp$$

(ii)

$$\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2 = q^2-2rp \quad \text{—— (1)}$$
$$(\alpha^2\beta^2)(\alpha^2\gamma^2)+(\alpha^2\beta^2)(\beta^2\gamma^2)+(\alpha^2\gamma^2)(\beta^2\gamma^2)$$
$$= (\alpha\beta\gamma)^2(\alpha^2+\beta^2+\gamma^2)$$
$$= r^2\left[(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)\right]$$
$$= r^2(p^2-2q) \quad \text{—— (2)}$$

$$\alpha^2\beta^2.\alpha^2\gamma^2.\beta^2\gamma^2$$
$$= (\alpha\beta\gamma)^4$$
$$= r^4$$

$\therefore$ polynomial is

$$x^3 \cancel{\mp} (q^2-2rp)x^2+r^2(p^2-2q)x-r^4=0$$
$$\therefore x^3+(2rp-q^2)x^2+r^2(p^2-2q)x-r^4=0$$

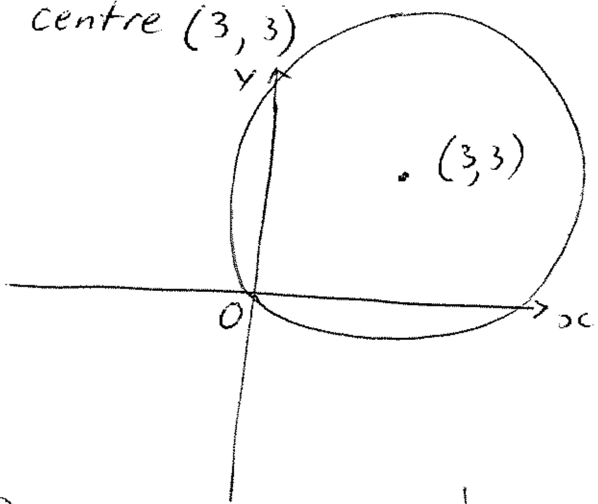
(c)

$$3\sqrt{(x-4)^2+(y-4)^2} = \sqrt{(x-12)^2+(y-12)^2}$$
$$= x^2-24x+144+y^2-24y+144$$
$$9(x^2-8x+16+y^2-8y+16) = x^2-24x+144+y^2-24y+144$$
$$9x^2-72x+\cancel{144}+9y^2-72y+\cancel{144} = x^2-24x+\cancel{144}+y^2-24y+\cancel{144}$$
$$8x^2-48x+8y^2-48y=0$$
$$x^2-6x+9+y^2-6y+9=18$$

$\therefore (x-3)^2 + (y-3)^2 = 18$

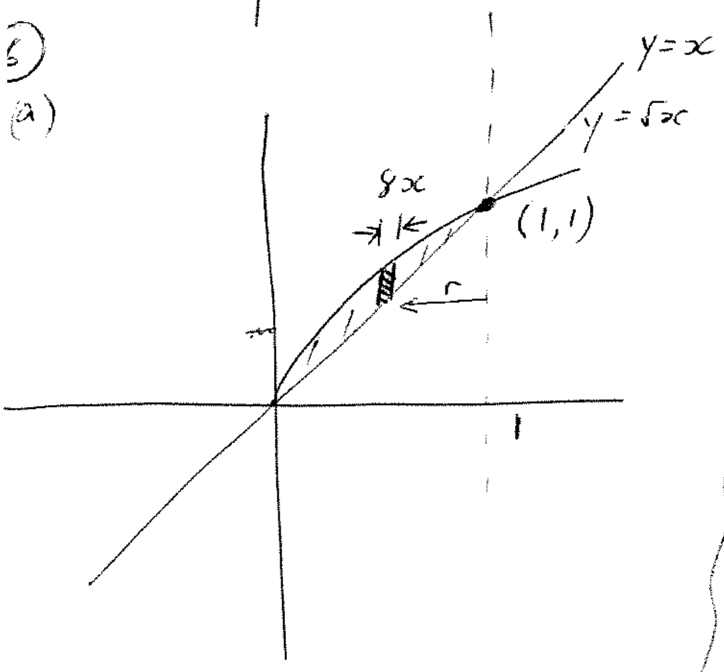
radius $\sqrt{18}$ or $3\sqrt{2}$

centre $(3,3)$




5)

(a)




$\delta V = 2\pi r h\, \delta x, \quad r = 1 - x$

$h = y_1 - y_2$

$= \sqrt{x} - x$

$\therefore \delta V = 2\pi(1-x)(\sqrt{x} - x)$

$V \doteq \sum_{x=0}^{x=1} 2\pi(1-x)(\sqrt{x}-x)$

$= 2\pi \int_0^1 \left(x^{\frac{1}{2}} - x - x^{\frac{3}{2}} + x^2\right) dx$

$= 2\pi \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2 - \frac{2}{5}x^{\frac{5}{2}} + \frac{1}{3}x^3\right]_0^1$

$= 2\pi\left(\frac{2}{3} - \frac{1}{2} - \frac{2}{5} + \frac{1}{3}\right)$

$= \frac{\pi}{5}$ units$^3$

(b)(i) $\Delta$s $COD$, $FHD$ are similar

$\therefore \frac{FH}{CO} = \frac{HD}{OD}$

$\frac{FH}{a} = \frac{h-x}{h}$

$FH = \frac{a}{h}(h-x)$

(ii) $C'F = C'H - FH$

$= a - \frac{a}{h}(h-x)$

$= a - a + \frac{ax}{h}$

$= \frac{ax}{h}$

$\cos H\hat{C'}G = \frac{C'F}{a}$

$= \frac{\frac{ax}{h}}{a}$

$= \frac{x}{h}$

$\therefore \frac{x}{h} = \cos^{-1} H\hat{C'}G$

(iii) $Area = \frac{1}{2}r^2(\theta - \sin\theta)$

$r = a, \ \theta = 2 . H\hat{C'}G$

$= 2\cos^{-1}\frac{x}{h}$

$Area = \frac{1}{2}a^2\left(2\cos^{-1}\frac{x}{h} - \sin\left(2\cos^{-1}\frac{x}{h}\right)\right)$

$= \cancel{\frac{1}{2}}a^2\left(\cancel{2}\cos^{-1}\frac{x}{h} - \cancel{2}\sin\left(\cos^{-1}\frac{x}{h}\right).\frac{x}{h}\right)$

$= a^2\left(\cos^{-1}\frac{x}{h} - \sqrt{\frac{h^2-x^2}{h^2}} . \frac{x}{h}\right)$

$= a^2\left(\cos^{-1}\frac{x}{h} - \frac{x}{h}\sqrt{1-\left(\frac{x}{h}\right)^2}\right)$

(iv)

$V = \int_0^h \left(a^2\cos^{-1}\frac{x}{h} - \frac{x}{h}\sqrt{1-\left(\frac{x}{h}\right)^2}\right) dx$

Let $\theta = \frac{x}{h}$, $\therefore h\, d\theta = dx$

(ii) cont

$$V = a^2 h \int_0^1 \left(\cos^{-1}\theta - \theta\sqrt{1-\theta^2}\right) d\theta$$

$$= a^2 h \left[\theta\cos^{-1}\theta - \sqrt{1-\theta^2} + \frac{1}{3}\left(1-\theta^2\right)^{\frac{3}{2}}\right]_0^1$$

$$= a^2 h\left[0 - 0 + 0 - \left(0 - 1 + \frac{1}{3}\right)\right]$$

$$= \frac{2a^2 h}{3} \text{ cubic units.}$$

(7) (a)

$$\int \frac{dx}{x^2 + qx + r}$$

$$= \int \frac{dx}{\left(x+\frac{q}{2}\right)^2 + r - \left(\frac{q}{2}\right)^2}$$

$$= \int \frac{dx}{\left(x+\frac{q}{2}\right)^2 + \frac{4r-q^2}{4}}$$

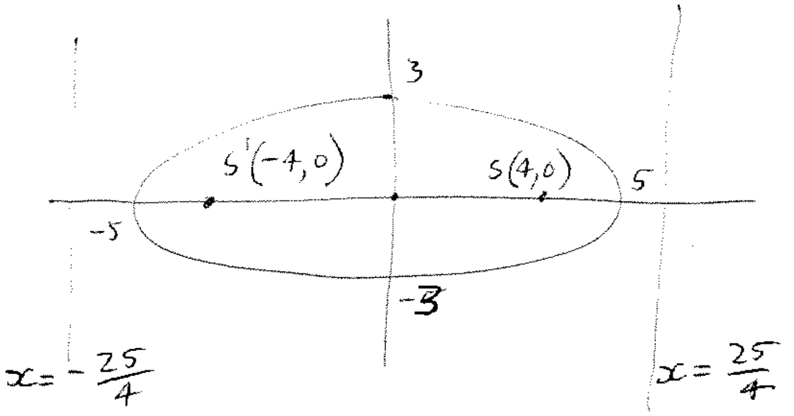
$$= \frac{1}{\sqrt{\frac{4r-q^2}{4}}} \tan^{-1}\left(\frac{x+\frac{q}{2}}{\sqrt{\frac{4r-q^2}{4}}}\right) + C$$

$$= \frac{2}{\sqrt{4r-q^2}} \tan^{-1} \frac{2x+q}{\sqrt{4r-q^2}} + C$$

only if $4r - q^2 > 0$

$$r > \frac{q^2}{4}$$

(b)



(ii) $\dfrac{2x}{25} + \dfrac{2y\frac{dy}{dx}}{9} = 0$, at $P(x_1, y_1)$

$$\frac{dy}{dx} = \frac{-9x_1}{25y_1}$$

Equation of tangent:

$$y - y_1 = \frac{-9x_1}{25y_1}(x - x_1)$$

$$9xx_1 + 25yy_1 = 9x_1^2 + 25y_1^2$$

$$= 225\left(\frac{x_1^2}{25} + \frac{y_1^2}{9}\right)$$

$$= 225$$

(iii) $m_{PQ} = \dfrac{y_1 - y_2}{x_1 - x_2}$

Equation PQ:

$$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1)$$

at $S(4,0)$

$$-y_1 = \frac{y_1 - y_2}{x_1 - x_2}(4 - x_1)$$

$$\cancel{-x_1y_1} + x_2y_1 = 4(y_1 - y_2) \cancel{- x_1y_1} + x_1y_2$$

$$4(y_1 - y_2) = x_2y_1 - x_1y_2$$

$$\therefore \; 4(y_2 - y_1) = x_1y_2 - x_2y_1$$

(iv) tangent at P:

$$9xx_1 + 25yy_1 = 225 \quad \text{——} \; (1)$$

tangent at Q:

$$9xx_2 + 25yy_2 = 225 \quad \text{——} \; (2)$$

$(1) \times y_2 - (2) \times y_1$

$$9x(x_1y_2 - x_2y_1) = 225(y_2 - y_1)$$

Now, from (iii)

$$x_1y_2 - x_2y_1 = 4(y_2 - y_1)$$

$$\therefore \; 9x \cdot 4(y_2 - y_1) = 225(y_2 - y_1)$$

$$36x = 225, \quad y_2 \neq y_1$$

$$x = \frac{225}{36} = \frac{25}{4}$$

$\therefore$ tangents intersect on directrix $x = \frac{25}{4}$.

(v) Equation of normal:

$$y - y_1 = \frac{25y_1}{9x_1}(x - x_1)$$

$$9x_1y - 9x_1y_1 = 25y_1x - 25x_1y_1$$

$$25y_1x - 9x_1y = 16x_1y_1$$

If normal passes through $S(4,0)$

$$100y_1 - 0 = 16x_1y_1$$

$$y_1(100 - 16x_1) = 0$$

$$y_1 = 0 \quad \text{or} \quad x_1 = \frac{100}{16} = \frac{25}{4}$$

but $|x_1| \leq 5$ (semi-major axis of ellipse)

$\therefore$ $y_1 = 0$ is the only solution

If normal passes through $S'(-4,0)$

$$-100y_1 = 16x_1y_1$$

$$y_1(100 + 16x_1) = 0$$

$$\therefore \; y_1 = 0 \quad \text{or} \quad x_1 = \frac{100}{16}$$

but $|x_1| \leq 5$

$\therefore$ $y = 0$ is the only solution

$\therefore$ Normals from $(5,0)$ and $(-5,0)$ pass through $S(4,0)$ and $S'(-4,0)$

(8) (a) $\lim_{x \to 0} \dfrac{1-\sqrt{1-x}}{x} \times \dfrac{1+\sqrt{1-x}}{1+\sqrt{1-x}}$

$= \lim_{x\to 0} \dfrac{1-(1-x)}{x(1+\sqrt{1-x})}$

$= \lim_{x\to 0} \dfrac{\cancel{x}}{\cancel{x}(1+\sqrt{1-x})}$

$= \dfrac{1}{1+\sqrt{1-0}}$

$= \dfrac{1}{2}$

(b) $P$(equal no. of heads on 1st 3 throws and last 3 throws)

$= P$(No heads on each set of 3 throws) $+ P$(1 head on each set of 3 throws)

$+ P$(2 heads on each set of 3 throws) $+ P$(3 heads on each set of 3 throws)

$= \left({}^3C_0\left(\frac{1}{2}\right)^3\right)\cdot\left({}^3C_0\left(\frac{1}{2}\right)^3\right) + \left({}^3C_1\,\frac{1}{2}\cdot\left(\frac{1}{2}\right)^2\right)\left({}^3C_1\,\frac{1}{2}\cdot\left(\frac{1}{2}\right)^2\right) + \left({}^3C_2\left(\frac{1}{2}\right)^2\cdot\frac{1}{2}\right) + \left({}^3C_2\left(\frac{1}{2}\right)^2\cdot\frac{1}{2}\right)$

$+ \left({}^3C_3\left(\frac{1}{2}\right)^3\right)\cdot\left({}^3C_3\left(\frac{1}{2}\right)^3\right)$

$= \left(\frac{1}{8}\right)^2 + \left(\frac{3}{8}\right)^2 + \left(\frac{3}{8}\right)^2 + \left(\frac{1}{8}\right)^2$

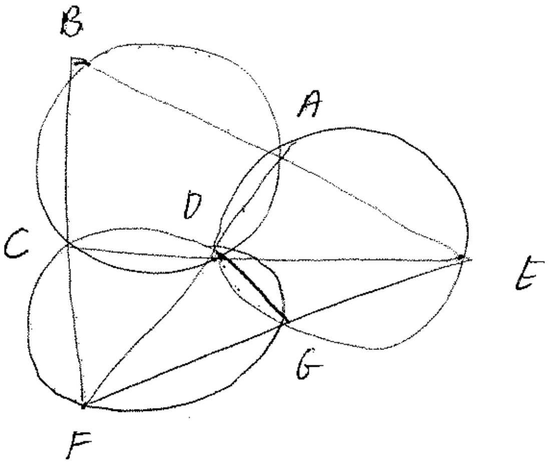
$= \dfrac{5}{16}$

$P$(More heads on 1st 3 throws) $= P$(More heads on last 3 throws)

$= \dfrac{1}{2}\left(1-\dfrac{5}{16}\right)$

$= \dfrac{11}{32}$

(c)




Construction: $DG, FG, EG$

$\because FCDG$ Cyclic Quad

$\therefore B\hat{C}D = D\hat{G}F$ (ext $\angle$ of cyclic quad = opp int $\angle$)

Similarly $B\hat{A}D = \hat{D}GE$

Now $B\hat{C}D + B\hat{A}D = 180^\circ$ (opp $\angle$s of cyclic quad)

$\therefore D\hat{G}F + D\hat{G}E = 180^\circ$

$\therefore F\hat{G}E = 180^\circ$

$\therefore E, G, F$ are collinear.

(d) (i) $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$f(x) = x^3 - 2, \quad f'(x) = 3x^2$

$$\therefore x_{n+1} = x_n - \frac{x_n^3 - 2}{3x_n^2}$$

$$= \frac{3x_n^3 - x_n^3 + 2}{3x_n^2}$$

~~= 2(x~~

$$= \frac{2x_n^{\cancel{3}}}{3\cancel{x_n^2}} + \frac{2}{3x_n^2} = \frac{2x_n}{3} + \frac{2}{3x_n^2} = \frac{2}{3}\left(x_n + \frac{1}{x_n^2}\right)$$

(ii) $RHS = \dfrac{(x_n - \sqrt[3]{2})^2 (2x_n + \sqrt[3]{2})}{3x_n^2}$

$= \cancel{(2x_n^2 + 2x_n \cdot \sqrt[3]{2}}$

$= \dfrac{\left(x_n^2 - 2x_n \sqrt[3]{2} + (\sqrt[3]{2})^2\right)(2x_n + \sqrt[3]{2})}{3x_n^2}$

$= \dfrac{2x_n^3 - 4x_n^2 \cdot \sqrt[3]{2} + \cancel{(\sqrt[3]{2})^2 \cdot 2x_n} + x_n^2 \cdot \sqrt[3]{2} - \cancel{2x_n (\sqrt[3]{2})^2} + (\sqrt[3]{2})^3}{3x_n^2}$

$= \dfrac{2x_n^3 - 3x_n^2 \cdot \sqrt[3]{2} + 2}{3x_n^2}$

$= \dfrac{2x_n^{\cancel{3}}}{3\cancel{x_n^2}} - \dfrac{\cancel{3x_n^2} \cdot \sqrt[3]{2}}{\cancel{3x_n^2}} + \dfrac{2}{3x_n^2}$

$= \dfrac{2x_n}{3} - \sqrt[3]{2} + \dfrac{2}{3x_n^2}$

$= \dfrac{2}{3}\left(x_n + \dfrac{1}{x_n^2}\right) - \sqrt[3]{2}$

$= x_{n+1} - \sqrt[3]{2}$, from (i)

$= LHS$

(iii) $x_n > \sqrt[3]{2}$

$\therefore \dfrac{(x_n - \sqrt[3]{2})^2 (2x_n + \sqrt[3]{2})}{3x_n^2} < \dfrac{(x_n - \sqrt[3]{2})^2 (2x_n + x_n)}{3x_n^2}$

$= \dfrac{(x_n - \sqrt[3]{2})^2 \cdot \cancel{3x_n}}{\cancel{3x_n^2}}$

$= \dfrac{(x_n - \sqrt[3]{2})^2}{x_n}$

Now, $\sqrt[3]{2} > 1$

$\therefore x_n > 1$

$$\therefore \frac{\left(x_n - \sqrt[3]{2}\right)^2\left(2x_n + \sqrt[3]{2}\right)}{3x_n^2} < \frac{\left(x_n - \sqrt[3]{2}\right)^2}{1}$$

$$\therefore x_{n+1} - \sqrt[3]{2} < \left(x_n - \sqrt[3]{2}\right)^2 \qquad \text{from (ii)}$$

(iv) Let $n = 1, 2, 3, \ldots 11.$

$$x_2 - \sqrt[3]{2} < \left(x_1 - \sqrt[3]{2}\right)^2 \qquad \text{from (iii)}$$

$$x_3 - \sqrt[3]{2} < \left(x_2 - \sqrt[3]{2}\right)^2 \qquad \text{"}$$

$$x_4 - \sqrt[3]{2} < \left(x_3 - \sqrt[3]{2}\right)^2 \qquad \text{"}$$

$$\vdots$$

$$x_{12} - \sqrt[3]{2} < \left(x_{11} - \sqrt[3]{2}\right)^2 \qquad \text{"}$$

$$\therefore x_{12} - \sqrt[3]{2} < \left(x_{11} - \sqrt[3]{2}\right)^2 < \left(x_{10} - \sqrt[3]{2}\right)^4 < \left(x_9 - \sqrt[3]{2}\right)^8 < \ldots$$

$$< \left(x_2 - \sqrt[3]{2}\right)^{2^{10}} < \left(x_1 - \sqrt[3]{2}\right)^{2^{11}}$$

$$\therefore x_{12} - \sqrt[3]{2} < \left(x_1 - \sqrt[3]{2}\right)^{2^{11}}$$

Now, $x_1 = 2$

$$\therefore x_{12} - \sqrt[3]{2} < \left(2 - \sqrt[3]{2}\right)^{2048}$$

$$x_{12} - \sqrt[3]{2} < \left(\left(2 - \sqrt[3]{2}\right)^{64}\right)^{32}$$

$$x_{12} - \sqrt[3]{2} < \left(4 \times 10^{-9}\right)^{32}$$

$$x_{12} - \sqrt[3]{2} < 4^{32} \times 10^{-288}$$

$$x_{12} - \sqrt[3]{2} < 1.85 \times 10^{19} \times 10^{-288} \qquad \text{(by calculator)}$$

$$x_{12} - \sqrt[3]{2} < 1.85 \times 10^{-269}$$

$\therefore x_{12}, \sqrt[3]{2}$ agree to at least 267 places